

MONODROMY FILTRATIONS AND THE TOPOLOGY OF TROPICAL VARIETIES

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ABSTRACT. We study the topology of tropical varieties that arise from a certain natural class of varieties. Our main tool is a theory of tropical degenerations that is a nonconstant coefficient analogue of Tevelev’s theory of tropical compactifications; we use this theory to construct simple normal crossings degenerations of a subvariety X of a torus, under mild hypotheses on X . These degenerations allow us to construct a natural, “multiplicity-free” parameterization of $\text{Trop}(X)$ by a topological space Γ_X . We give a geometric interpretation of the cohomology of Γ_X in terms of the action of a monodromy operator on the cohomology of X . This gives bounds on the Betti numbers of Γ_X in terms of the Betti numbers of X . When X is a sufficiently general complete intersection, this allows us to show that the cohomology of $\text{Trop}(X)$ vanishes in degree less than $\dim X$. In addition, we give a description for the top power of the monodromy operator acting on middle cohomology in terms of the volume pairing on Γ_X .

1. INTRODUCTION

Let \mathcal{O} be a discrete valuation ring with field of fractions K . Tropicalization is a procedure which takes as input a sub-variety of an algebraic torus over K $X \subset (K^*)^n$, and associates to it a balanced weighted rational polyhedral complex $\text{Trop}(X) \subseteq \mathbb{R}^n$. Several questions naturally arise in this framework; for instance, one may ask what combinatorial properties of $\text{Trop}(X)$ correspond to geometric properties of X . One may also ask what constraints being a tropicalization places on the topology of a polyhedral complex. In [H], Hacking proved that if X is a subvariety of $(\mathbb{C}^*)^n$ satisfying a certain genericity condition, then the link of the fan $\text{Trop}(X)$ only has reduced rational homology in the top dimension. Hacking’s result holds for a number of examples, including generic intersections of ample hypersurfaces in projective toric varieties. In [Sp2, Sec. 10], Speyer showed that if C is a genus g curve in $(K^*)^n$ satisfying a genericity condition then there exists a balanced metric graph Γ with $b_1(\Gamma) \leq g$ and a parameterization $i : \Gamma \rightarrow \text{Trop}(C)$ that is affine-linear on edges. Our results can be seen as the analogue of Hacking’s result for varieties defined over K or as a higher-dimensional generalization of Speyer’s result.

All of our results require that the variety X be *schön*, a natural condition introduced in [T] and generalized by [Q] to the nonconstant coefficient case. This condition means that the ambient torus $(K^*)^n$ may be compactified to a toric scheme \mathbb{P} over an extension of $\text{Spec } \mathcal{O}$ such that the intersection of $\mathcal{X} = \overline{X} \subset \mathbb{P}$ with each open torus orbit U_P is smooth of the expected dimension. For appropriate \mathbb{P} , \mathcal{X} is then a simple normal crossings degenerations of X (c.f. 3.9).

The existence of such simple normal crossings degenerations for a schön X allows us to construct a natural “parameterizing space” Γ_X . This generalizes a construction introduced by Speyer [Sp2] when X has dimension 1. The space Γ_X is closely related to the dual complex of an appropriate degeneration \mathcal{X} of X ; in this guise it already appears implicitly in [H], as well as in [HKT]. Kontsevich-Soibelman use rigid analytic techniques to construct a similar polyhedral complex, with an integral affine structure, from a suitable degeneration of X in [KS].

The space Γ_X we construct is *independent* of a choice of model \mathcal{X} for X ; it depends only on X and its embedding in the torus. Moreover, Γ_X comes equipped with a canonical map to $\text{Trop}(X)$; a choice of sufficiently fine triangulation of $\text{Trop}(X)$ gives Γ_X the structure of a polyhedral complex. When Γ_X is viewed in such a way, the natural parameterization $\Gamma_X \rightarrow \text{Trop}(X)$ is affine-linear on polyhedra. This parameterization has several nice properties. For instance, it is natural under monomial morphisms: if X and Y are schön subvarieties of tori T and T' and $\phi : T \rightarrow T'$ is a homomorphism taking X to Y , then there is an induced map of complexes $\Gamma_X \rightarrow \Gamma_Y$ that commutes with parameterizations. Moreover, Γ_X satisfies a balancing condition analogous to the one satisfied by all tropical varieties. Finally, it is “not far” from $\text{Trop}(X)$: if the intersections of \mathcal{X} with open torus orbits U_P in \mathbb{P} satisfy certain connectedness hypotheses, we may equate the cohomology of Γ_X and $\text{Trop}(X)$ in certain degrees. We hope that parameterizing complexes will be seen as a fundamental object in tropical geometry.

Our main results (principally Theorem 6.1 and Corollary 6.3) relate the cohomology of Γ_X to geometric invariants of X . In particular we consider the étale cohomology $H_{\text{ét}}^*(X_{K^{\text{sep}}}, \mathbb{Q}_l)$; this cohomology comes equipped with a natural filtration, called the weight filtration. We construct a natural isomorphism between the “weight 0” subquotient $W_0 H_{\text{ét}}^r(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ arising from this filtration and the cohomology $H^r(\Gamma_X, \mathbb{Q}_l)$ of Γ_X . We then use this to show that if X is the generic intersection of ample hypersurfaces in a toric scheme \mathbb{P} , then $H^r(\text{Trop}(X), \mathbb{Q}_l)$ vanishes for $1 \leq r < \dim X$, a non-constant coefficient analog of Hacking’s result.

The main tool we use is the Rapoport-Zink weight spectral sequence [RZ]. Under the schön condition, after a finite base-extension \mathcal{O}' of \mathcal{O} , we may compactify the ambient torus to a toric scheme \mathbb{P} defined over \mathcal{O}' so that the central fiber of the closure \mathcal{X} of X in \mathbb{P} is a divisor with simple normal crossings. The divisor, a degeneration of X , has a stratification coming from intersections of its irreducible components. The Rapoport-Zink spectral sequence then gives a very explicit formula for the cohomology on X , together with its weight filtration, in terms of these strata. The E_1 -term of the weight spectral sequence is formed from the cohomology groups of the strata with boundary maps built from the data of restriction maps and Gysin maps. The spectral sequence converges to the cohomology of the general fiber, and the induced filtration is the weight filtration. Moreover, the weight spectral sequence degenerates at E_2 . We thus obtain an explicit description of the smallest nontrivial piece of the filtration which is isomorphic to the cohomology groups of Γ_X .

It is interesting to compare this result to results of Berkovich [B] on rigid analytic spaces. In particular, Berkovich shows that the cohomology group $H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r}$ arising in our result is isomorphic to the cohomology of the Berkovich space X^{an} attached to X . Our result thus suggests a strong link between Γ_X and X^{an} . In fact, Speyer [Sp1] constructs a natural map from X^{an} to $\text{Trop}(X)$. This map factors

through the map $\Gamma_X \rightarrow \text{Trop}(X)$, and it is natural to ask if the resulting map $X^{\text{an}} \rightarrow \Gamma_X$ map is a homotopy equivalence. Links between tropical geometry and rigid geometry have also appeared in works of Einsiedler-Kapranov-Lind [EKL], and Payne [P].

Under additional hypotheses, one can relate the results above to questions involving monodromy. A variety defined over $\text{Spec } K$ is analogous to a family of varieties defined over a punctured disc. The fundamental group of the punctured disc acts on the cohomology of a general fiber of such a family by monodromy. The analogue of this monodromy action for varieties over $\text{Spec } K$ is the action of the inertia group I_K of K on the étale cohomology $H_{\text{ét}}^*(X_{K^{\text{sep}}}, \mathbb{Q}_l)$. After a possible finite base-extension of \mathcal{O} , this action is unipotent, and is given by the *monodromy operator*

$$N : H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l(-1))$$

an endomorphism of the étale cohomology that is essentially the (matrix) logarithm of the action. (We refer the reader to section 7 for precise definitions). The action of N induces an increasing filtration on the cohomology. The weight-monodromy conjecture asserts that this filtration coincides (up to a shift in degree) with the weight filtration described above. Although it is not completely settled, this conjecture is known to be true in many cases of interest; for instance, it is known X is a curve, surface, or an abelian variety. Ito [I] has proven the weight-monodromy conjecture when \mathcal{O} has equal characteristic. Thus, in these situations, one can interpret Theorem 6.1 as an isomorphism between the cohomology of Γ_X and the smallest nontrivial piece of the monodromy filtration of the cohomology of \overline{X}_K , the closure of X_K in the generic fiber of \mathbb{P} :

$$H^r(\Gamma_X, \mathbb{Q}_l) \cong H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r}.$$

As a consequence, Corollary 7.3 gives a generalization of Speyer's result, bounding the Betti numbers of Γ_X in terms of those of X :

$$b_r(\Gamma_X) \leq \frac{1}{r+1} b_r(X).$$

In Proposition 7.4, we give an interpretation of the top power of monodromy operator acting on the middle-dimensional cohomology

$$N^n : H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_n / H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{n-1} \rightarrow H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{-n}(-n)$$

using the isomorphisms

$$\begin{aligned} H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_n / H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{n-1} &\cong H_n(\Gamma, \mathbb{Q}_l)(-n) \\ H_{\text{ét}}^n(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r} &\cong H^n(\Gamma_X, \mathbb{Q}_l). \end{aligned}$$

The operator can be viewed as a bilinear pairing on $H^n(\Gamma_X, \mathbb{Q}_l)$ in which case it is the volume pairing that takes a pair of top-dimensional cycles to the oriented volume of their intersection. This specializes to the length pairing in the case of curves. In the case of genus 1 curves, by a straightforward application of the conductor discriminant formula, we are able to recover the spatial schön analog of a result proved by the second-named author with Hannah Markwig and Thomas Markwig [KMM]: the valuation of the j -invariant of an elliptic curve X with potentially multiplicative reduction is equal to $-a$ where a is the length of the unique cycle in Γ_X .

Our arguments are very similar to those of Hacking and Speyer. Hacking uses a spectral sequence coming from a weight filtration on a complex of differential forms while we use the Rapoport-Zink spectral sequence. Speyer's results use a resolution of the structure sheaf of a degeneration of C coming from a stratification induced by a toric scheme.

We should mention the related results of Gross and Siebert [GS]. There, the authors construct a scheme X_0 from an integer affine manifold and a toric polyhedral decomposition. If X_0 is embedded in a family \mathcal{X} over $\mathbb{C}[[t]]$, they are able to determine the limiting mixed Hodge structure in terms of the combinatorial data.

We would like to thank Brian Conrad, Richard Hain, Kalle Karu, Sean Keel, Sam Payne, Zhenhua Qu, Bernd Siebert, David Speyer, Alan Stapledon, and Bernd Sturmfels for valuable discussions.

2. TORIC SCHEMES

We begin by reviewing a construction that attaches a degenerating family of toric varieties over a discrete valuation ring to a rational polyhedral complex in \mathbb{R}^n . This has appeared several times in the literature [Sp1] [NS] [S]. We follow the approach of [NS] here. Fix a discrete valuation ring \mathcal{O} , with field of fractions K and residue field k , and a uniformizer π of \mathcal{O} .

Definition 2.1. A rational polyhedral complex in \mathbb{R}^n is a collection Σ of finitely many convex rational polyhedra $P \subset \mathbb{R}^n$ with the following properties:

- If $P \in \Sigma$ and P' is a face of P , then P' is in Σ .
- If $P, P' \in \Sigma$ then $P \cap P'$ is a face of both P and P' .

Given a Σ as above, we can construct a fan $\tilde{\Sigma}$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ as follows: for each $P \in \Sigma$ let \tilde{P} be the closure in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ of the set

$$\{(x, a) \in \mathbb{R}^n \times \mathbb{R}_{> 0} : \frac{x}{a} \in P\}.$$

Then \tilde{P} is a rational polyhedral cone in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Its facets come in two types:

- cones of the form \tilde{P}' , where P' is a facet of P , and
- the cone $P_0 = \tilde{P} \cap (\mathbb{R}^n \times \{0\})$, which is the limit as a goes to zero of the polyhedron aP in \mathbb{R}^n .

We let $\tilde{\Sigma}$ be the collection of cones of the form \tilde{P} and P_0 for P in Σ . It is a rational polyhedral fan in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Note that $\Sigma = \tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$. On the other hand the fan Σ_0 given by $\tilde{\Sigma} \cap (\mathbb{R}^n \times \{0\})$ is the limit as a approaches zero of the polyhedral complexes $a\Sigma$.

Remark 2.2. In fact, the association $\Sigma \mapsto \tilde{\Sigma}$ defines a bijection between the set of polyhedral complexes in \mathbb{R}^n and the set of fans in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ for which every cone contained in $\mathbb{R}^n \times \{0\}$ is the boundary of a cone that meets $\mathbb{R}^n \times \mathbb{R}_{> 0}$. Its inverse is $\tilde{\Sigma} \mapsto \tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$.

Let $X(\tilde{\Sigma})_{\mathbb{Z}}$ be the toric scheme over \mathbb{Z} associated to the fan $\tilde{\Sigma}$. (The construction associating a toric variety to a fan is usually given over a field, but works just as well with coefficients in \mathbb{Z} .) Projection from $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ induces a map of fans from $\tilde{\Sigma}$ to the fan $\{0, \mathbb{R}_{\geq 0}\}$ associated to $\mathbb{A}_{\mathbb{Z}}^1$. This gives rise to a map of toric varieties $\pi_{\mathbb{Z}} : X(\tilde{\Sigma})_{\mathbb{Z}} \rightarrow \mathbb{A}_{\mathbb{Z}}^1$. As remarked in [NS], this map is flat and torus equivariant. Let $\iota : \text{Spec } \mathcal{O} \rightarrow \mathbb{A}_{\mathbb{Z}}^1$ be the map corresponding to the map $\mathbb{Z}[t] \rightarrow \mathcal{O}$

that takes t to π . We let $X(\tilde{\Sigma})$ be the scheme over \mathcal{O} obtained by base change from $X(\tilde{\Sigma})_{\mathbb{Z}}$ via ι , and let $\pi : X(\tilde{\Sigma}) \rightarrow \mathcal{O}$ be the projection.

We summarize results of [NS] concerning this construction:

- The general fiber $X(\tilde{\Sigma}) \times_{\text{Spec } \mathcal{O}} \text{Spec } K$ is isomorphic to the toric variety over K associated to Σ_0 .
- If Σ is *integral*, i.e. the vertices of every polyhedron in Σ lie in \mathbb{Z}^n , then the special fiber $X(\tilde{\Sigma})_k = X(\tilde{\Sigma}) \times_{\text{Spec } \mathcal{O}} \text{Spec } k$ is reduced.
- There is an inclusion-reversing bijection between closed torus orbits in $X(\tilde{\Sigma})_k$ and polyhedra P in Σ ; the irreducible components of $X(\tilde{\Sigma})_k$ correspond to vertices in Σ ; the intersection of a collection of irreducible components corresponds to the smallest polyhedron in Σ containing all of their vertices.

Note that adjoining a d th root of π to \mathcal{O} has the effect of rescaling Σ by d ; that is, if $\mathcal{O}' = \mathcal{O}[\pi^{\frac{1}{d}}]$, then the base change of the family $X(\tilde{\Sigma}) \rightarrow \mathcal{O}$ is the family $X(\widetilde{d\Sigma}) \rightarrow \mathcal{O}'$. In particular, given any toric scheme coming from a polyhedral complex Σ , we can choose d such that $d\Sigma$ is integral; after taking a suitable ramified base change of \mathcal{O} the special fiber of the family $X(\tilde{\Sigma}) \rightarrow \mathcal{O}$ will be reduced.

We will be particularly interested in degenerations of toric varieties in which the special fiber is a divisor with simple normal crossings. These are easy to construct, because the boundary of a smooth toric variety is always a divisor with simple normal crossings:

Proposition 2.3. *Let Σ be a rational polyhedral complex in \mathbb{R}^n . There exists an integer d , and a subdivision Σ' of $d\Sigma$ such that the general fiber of the scheme $X(\tilde{\Sigma}')$ is a smooth toric variety and the special fiber of $X(\tilde{\Sigma}')$ is a divisor with simple normal crossings. Moreover, if the recession fan Σ_0 is already simplicial and unimodular, Σ' can be chosen to have $\Sigma'_0 = \Sigma_0$.*

Proof. Choose an integer l_1 such that $l_1\Sigma$ is integral. Fulton [F, Sec. 2.6] gives an algorithm for constructing a subdivision $\tilde{\Sigma}'$ of the fan $\widetilde{l_1\Sigma}$ such that all the cones of $\tilde{\Sigma}'$ are simplicial and unimodular. Pick an integer l_2 sufficiently divisible so $\Sigma_1 = \tilde{\Sigma}' \cap (\mathbb{R}^n \times \{l_2\})$ is integral. Σ_1 is a subdivision of \mathbb{R}^n with the property that each of its recession cones is simplicial. Because $\tilde{\Sigma}'$ is simplicial each of its cones is of the form $\tilde{P} + Q_0$ where P is a simplex in Σ_1 , Q_0 is a cone in the recession fan $(\Sigma_1)_0$, and $\tilde{P} \cap Q_0 = \{0\}$. Consequently, the corresponding polyhedron of Σ_1 is $(\tilde{P} + Q_0) \cap (\mathbb{R}^n \times \{l_2\}) = P + Q_0$. For a polyhedron F in \mathbb{R}^k , let $N_F = \mathbb{Z}^k \cap \text{Span}_{\mathbb{R}}(F - w)$ where w is a point of F . If F is a rational polytope, then N_F has the property a basis of it can be extended to a basis of \mathbb{Z}^k .

We claim $N_P + N_{Q_0} = N_{P+Q_0}$ for every cone $\tilde{P} + Q_0$ of $\tilde{\Sigma}'$. It is clear that $N_P + N_{Q_0} \subseteq N_{P+Q_0}$. The inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1} = \mathbb{Z}^n \times \mathbb{Z}$ given by $x \mapsto (x, l_2)$ identifies N_{P+Q_0} with the intersection of $N_{\tilde{P}+Q_0}$ with $\mathbb{Z}^n \times \{l_2\}$. Since $\tilde{P} + Q_0$ is unimodular, $N_{\tilde{P}} + N_{Q_0}$ is equal to $N_{\tilde{P}+Q_0}$. Therefore, if $x \in N_{P+Q_0}$, $(x, l_2) = x_{\tilde{P}} + x_{Q_0}$ where $x_{\tilde{P}} \in \tilde{P}$ and $x_{Q_0} \in Q_0$. Since the last coordinate of x_{Q_0} is 0, $x_{\tilde{P}} = (x_P, l_2)$ for $x_P \in N_P$. It follows that $x = x_P + x_{Q_0}$.

Let Σ_1^b be the union of the bounded polyhedra of Σ_1 . By an important step of the proof of semi-stable reduction [KKMS, Ch. 3, Thm 4.1], there is an integer l_2 and a unimodular triangulation $\Sigma_1^{b'}$ of $l_2\Sigma_1^b$. This induces a subdivision of $l_2\Sigma_1$ where the polyhedra whose relative interior is contained in the relative interior of

$l_2(P + Q_0)$ are of the form $P' + Q_0$ where P' is a simplex in $\Sigma_1^{b'}$ whose relative interior is contained in the relative interior of l_2P . We call this subdivision Σ' . It is simplicial by construction. We claim that it is also unimodular. It suffices to show that maximal cones in $\tilde{\Sigma}'$ are unimodular. Let $\tilde{P}' + Q_0$ be a maximal cone in $\tilde{\Sigma}'$. Then the relative interior of P' is contained in the relative interior of l_2P with $\dim P = \dim P'$. Since P' is unimodular, its \mathbb{Z} -affine span is $N_{P'}$. Consequently, since $N_{P'} + N_{Q_0} = N_P + N_{Q_0} = N_{P+Q_0}$, we see that any element of N_{P+Q_0} can be written as integer combination $\sum m_i v_i + \sum n_i w_i$ where v_i are vertices of P' , w_i are the primitive vectors along the rays of Q_0 , and $\sum m_i = 1$. Consequently, any element of $N_{P+Q_0} \times \{1\}$ can be written as an integer combination of the primitive vectors along the rays of $\tilde{P}' + Q_0$. Consequently these vectors generate $N_{\tilde{P}'+Q_0} \subset \mathbb{Z}^{n+1}$. Therefore $\tilde{P}' + Q_0$ is smooth.

If Σ_0 was simplicial and unimodular to begin with, none of these steps would have affected the cones Q_0 of Σ_0 .

The upshot is that $X(\tilde{\Sigma}')$ is a smooth toric variety with a birational morphism $X(\tilde{\Sigma}') \rightarrow X(\tilde{\Sigma})$. The induced map $X(\tilde{\Sigma}') \rightarrow \mathcal{O}$ is the toric scheme associated to the integral polyhedral complex Σ' . The general fiber of $X(\tilde{\Sigma}')$ over \mathcal{O} corresponds to the fan Σ'_0 , and is therefore smooth. The special fiber is a union of irreducible components of the boundary of $X(\tilde{\Sigma}')$, and is therefore a divisor of $X(\tilde{\Sigma}')$ with simple normal crossings. \square

3. TROPICAL DEGENERATIONS

We now describe the applications of tropical geometry to the study of degenerations of varieties over K . These techniques have their origins in the Speyer's thesis [Sp1]. The approach we take here is due to Tevelev [T] in the “constant coefficient case”; the extension of Tevelev's work to the case of an arbitrary DVR done by Zhenhua Qu in part of his Ph.D. thesis [Q].

Let \overline{K} be an algebraic closure of K . There is a unique valuation

$$\text{ord} : \overline{K} \rightarrow \mathbb{Q}$$

such that $\text{ord}(\pi) = 1$.

Let $\mathcal{T} \cong \mathbb{G}_m^n$ be a split n -dimensional torus over \mathcal{O} , and let $T = \mathcal{T} \times_{\mathcal{O}} K$ be the corresponding torus over K . The valuation ord gives rise to a map

$$\text{val} : \mathcal{T}(\overline{K}) \rightarrow \mathbb{Q}^n,$$

by fixing an isomorphism of \mathcal{T} with \mathbb{G}_m^n (and hence an isomorphism of $\mathcal{T}(\overline{K})$ with $(\overline{K}^\times)^n$.) Let X be a closed subvariety of T , defined over K .

Definition 3.1 ([EKL], 1.2.1). : The tropical variety $\text{Trop}(X)$ associated to X is the closure of $\text{val}(X(\overline{K}))$ in \mathbb{R}^n .

Given such an X , one can ask for a well-behaved compactification \overline{X} of X , and a well-behaved degeneration of \overline{X} over \mathcal{O} . The problem of finding such a degeneration is intimately connected to the set $\text{Trop}(X)$.

Let Σ be a rational polyhedral complex in \mathbb{R}^n , and let \mathbb{P} be the corresponding toric scheme over \mathcal{O} . Identify the group of cocharacters of \mathcal{T} with \mathbb{Z}^n in \mathbb{R}^n ; this identifies \mathbb{T} with the open torus orbit on \mathbb{P} .

We can thus take the closure of \mathcal{X} of X in \mathbb{P} . By [Sp1], 2.4.1, the scheme \mathcal{X} is proper over \mathcal{O} if, and only if, $\text{Supp } \Sigma$ contains $\text{Trop}(X)$.

We assume henceforth that $\text{Supp } \Sigma$ contains $\text{Trop}(X)$. Let \overline{X} be the fiber of \mathcal{X} over K , and \mathcal{X}_k be the special fiber of \mathcal{X} . The natural multiplication map

$$\mathcal{T} \times_{\mathcal{O}} \mathbb{P} \rightarrow \mathbb{P}$$

restricts to a multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}.$$

Definition 3.2. The pair (X, \mathbb{P}) is *tropical* if the map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}$$

is faithfully flat, and $\mathcal{X} \rightarrow \mathcal{O}$ is proper.

We then have the following, due to Tevelev ([T], 2.5) in the “constant coefficient case”. In the general situation they are due to Qu ([Q], 3.1.8, 3.1.10).

Proposition 3.3. *Suppose (X, \mathbb{P}) is tropical and let $\mathbb{P}' \rightarrow \mathbb{P}$ be a morphism of toric schemes corresponding to a refinement Σ' of Σ . Then (X, \mathbb{P}') is also tropical.*

Proposition 3.4. *If (X, \mathbb{P}) is a tropical pair then $\text{Supp } \Sigma = \text{Trop}(X)$.*

Following Speyer ([Sp1], 2.4) If (X, \mathbb{P}) is a tropical pair, we call \overline{X} a *tropical compactification* of X , and \mathcal{X}_k a *tropical degeneration* of X .

The combinatorics of the special fiber of a tropical degeneration of X is closely related to the combinatorics of $\text{Trop}(X)$. In particular if (X, \mathbb{P}) is a tropical pair, and \mathcal{X} is the corresponding tropical degeneration, then a polyhedron P of Σ corresponds to the closure of a torus orbit in the special fiber of \mathbb{P} . Call this torus orbit closure \mathbb{P}_P . Then the intersection \mathcal{X}_P of \mathcal{X} with \mathbb{P}_P is a closed subscheme of \mathcal{X}_k . Moreover, if P and P' are polyhedra of Σ , and Q is the smallest polyhedron in Σ containing both P and P' , then the intersection of \mathcal{X}_P and $\mathcal{X}_{P'}$ is \mathcal{X}_Q .

Let U_P be the open torus orbit corresponding to P . Fix a point p in U_P , and consider the fiber over p of the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}.$$

On the one hand, $m^{-1}(p)$ is nonempty of dimension equal to the dimension of X , since m is flat and surjective. On the other hand, by projection onto \mathcal{X} , $m^{-1}(p)$ is isomorphic to the product $\mathcal{T}_P \times (\mathcal{X} \cap U_P)$, where \mathcal{T}_P is the subgroup of \mathcal{T}_0 that acts trivially on U_P . Since $(\mathcal{X} \cap U_P)$ is dense in \mathcal{X}_P we find that \mathcal{X}_P is nonempty of dimension equal to the dimension of X minus the dimension of P .

On the other hand, let w be a point in the relative interior of P . Then w corresponds to a cocharacter of T , and $w(\pi)$ specializes to a point p in U_P . Projection onto \mathcal{T} identifies $m^{-1}(p)$ with the mod π reduction in $_w X$ of $w(\pi)X$. (More formally, $\text{in}_w X$ can be defined as the special fiber of the closure in \mathcal{T} of the subscheme $w(\pi)X$ of T . Note that this depends only on X and T , not on our choice of Σ .) In particular we have

$$\text{in}_w X \cong \mathcal{T}_P \times (\mathcal{X} \cap U_P)$$

for any w in the relative interior of P . We have thus shown:

Lemma 3.5. *The space $\text{in}_w X$ is a torus bundle over $\mathcal{X} \cap U_P$. In particular, if $C(\text{in}_w X)$ is the set of connected components of $(\text{in}_w X)_{\overline{k}}$, and $C(\mathcal{X} \cap U_P)$ is the set of connected components of $(\mathcal{X} \cap U_P)_{\overline{k}}$, then the maps*

$$\text{in}_w X \cong m^{-1}(p) \rightarrow \mathcal{X} \cap U_P$$

give a natural bijection of $\mathbb{C}(\text{in}_w X)$ with $C(\mathcal{X} \cap U_P)$.

We will be particularly interested in tropical pairs (X, \mathbb{P}) where the multiplication map $m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}$ is *smooth*. This condition is due to Tevelev in the constant coefficient case, and Qu in general.

Definition 3.6. A subvariety X of \mathcal{T} is *schön* if there exists a tropical pair (X, \mathbb{P}) such that the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}$$

is smooth.

One then has ([Q], 3.1.10):

Proposition 3.7. *If X is schön, then for any tropical pair (X, \mathbb{P}) , the multiplication map*

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}$$

is smooth.

Note that if X is schön then it is smooth (consider the preimage of the identity in \mathbb{T} under the multiplication map.) In fact, we have:

Proposition 3.8. *The following are equivalent:*

- (1) X is schön.
- (2) $\text{in}_w X$ is smooth for all $w \in \text{Trop}(X)$.
- (3) For any tropical pair (X, \mathbb{P}) , and any polyhedron P in Σ , $\mathcal{X} \cap U_P$ is smooth.

Proof. Statements 2) and 3) are clearly equivalent since we have seen that $\text{in}_w X$ is the product of a torus with $\mathcal{X} \cap U_P$, where P is the polyhedron in Σ that contains w in its relative interior.

As for the equivalence of 1) and 2), fix a tropical pair (X, \mathbb{P}) . We have seen that the fibers of the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X} \rightarrow \mathbb{P}$$

are isomorphic to $\text{in}_w X$ as w ranges over $\text{Trop}(X)$. So 1) implies 2) is clear. For the converse, note that since m is faithfully flat, to show it is smooth it suffices to show that it has smooth fibers. \square

It is easy to construct tropical degenerations of schön varieties X in which the special fiber is a divisor with simple normal crossings. In particular we have:

Proposition 3.9 (c.f. [H], proof of 2.4). *Let X be schön. There exists an integer d and a tropical pair (X, \mathbb{P}) over $\mathcal{O}[\pi^{\frac{1}{d}}]$ such that \overline{X} is smooth over $K[\pi^{\frac{1}{d}}]$, and \mathcal{X}_k is a divisor in \mathcal{X} with simple normal crossings.*

Proof. Let (X, \mathbb{P}) be any tropical pair over \mathcal{O} , and let Σ be the rational polyhedral complex corresponding to \mathbb{P} . By Proposition 2.3, we can find a refinement Σ' of Σ and an integer d such that the corresponding toric scheme \mathbb{P}' (viewed over $\mathcal{O}[\pi^{\frac{1}{d}}]$) has smooth general fiber, and special fiber a divisor with simple normal crossings.

Then (X, \mathbb{P}') is also tropical, and the multiplication map

$$m : \mathcal{T} \times_{\mathcal{O}} \mathcal{X}' \rightarrow \mathbb{P}'$$

is smooth by the previous proposition. Since the special fiber of \mathbb{P}' is a divisor with simple normal crossings, so is the special fiber of $\mathcal{T} \times_{\mathcal{O}} \mathcal{X}'$. Hence the special fiber of \mathcal{X}' is a divisor with simple normal crossings as well. Similarly, the general fiber of \mathcal{X}' is smooth because the general fiber of \mathbb{P}' is smooth. \square

Definition 3.10. We call a pair (X, \mathbb{P}) of the sort produced by Proposition 3.9 a *normal crossings pair*. If (X, \mathbb{P}) is a normal crossings pair, and Σ is the polyhedral decomposition of $\text{Trop}(X)$ corresponding to \mathbb{P} , we say that Σ is a *normal crossings decomposition* of $\text{Trop}(X)$.

Remark 3.11. In much of what follows, we will often need to attach a normal crossings pair to a schön variety X over \mathcal{O} . To do this we may need to replace \mathcal{O} with a ramified extension $\mathcal{O}[\pi^{\frac{1}{d}}]$; this is harmless and we often do so without comment.

4. PARAMETERIZED TROPICAL VARIETIES

In this section, given a schön subvariety X of \mathcal{T} , we construct a natural parameterization of $\text{Trop}(X)$ by a topological space Γ_X . This parameterization is functorial in a sense we make precise below. Moreover, we will see in the next section that the space Γ_X encodes more precise information about the cohomology of X than $\text{Trop}(X)$ does. Our approach generalizes a construction of Speyer ([Sp2], proof of Theorem 10.8) when X has dimension 1.

Suppose we have a normal crossings pair (X, \mathbb{P}) so $\text{Supp}(\Sigma) = \text{Trop}(X)$. We associate to (X, \mathbb{P}) a polyhedral complex $\Gamma_{(X, \mathbb{P})}$ as follows: its k -cells are pairs (P, Y) , where P is a polyhedron in Σ and Y is an irreducible component of \mathcal{X}_P . The cells on the boundary of (P, Y) are the cells of the form (P_i, Y_i) , where P_i is a facet of P and Y_i is the irreducible component of \mathcal{X}_{P_i} containing Y (there is exactly one such irreducible component because \mathcal{X}_{P_i} is smooth, so its irreducible components do not meet). The complex $\Gamma_{(X, \mathbb{P})}$ maps naturally to Σ by sending (P, Y) to P .

Proposition 4.1. *The underlying topological space of $\Gamma_{(X, \mathbb{P})}$ depends only on X .*

Proof. Any two polyhedral decompositions of $\text{Trop}(X)$ have a common refinement; we can further refine this to be a normal crossings decomposition of $\text{Trop}(X)$. It thus suffices to show that if Σ and Σ' are normal crossings decompositions of $\text{Trop}(X)$, with associated normal crossings pairs (X, \mathbb{P}) and (X, \mathbb{P}') , and Σ' refines Σ , then the underlying topological spaces of $\Gamma_{(X, \mathbb{P})}$ and $\Gamma_{(X, \mathbb{P}')}$ are isomorphic.

Since Σ' is a refinement of Σ , we have a map $\mathbb{P}' \rightarrow \mathbb{P}$. Let \mathcal{X}' be the degeneration corresponding to the pair (X, \mathbb{P}') . If P is a polyhedron of Σ , and P' is a polyhedron of Σ' contained in P , then this map induces a map of $\mathcal{X}'_{P'}$ to \mathcal{X}_P . In particular, for every pair (P', Y') of $\Gamma_{(X, \mathbb{P}')}$, the image of Y' in \mathcal{X} is contained in a unique irreducible component Y of \mathcal{X}_P . The map taking (P', Y') to (P, Y) is then a map of polyhedral complexes $\Gamma_{(X, \mathbb{P}')} \rightarrow \Gamma_{(X, \mathbb{P})}$.

We have a commutative diagram:

$$\begin{array}{ccc} \Gamma_{(X, \mathbb{P}')} & \xrightarrow{\quad} & \text{Trop}(X) \\ \downarrow & \nearrow & \\ \Gamma_{(X, \mathbb{P})} & & \end{array}$$

The fiber of $\pi : \Gamma_{(X, \mathbb{P})} \rightarrow \text{Trop}(X)$ over a point w is in canonical bijection with the set $C(\mathcal{X}_P)$ of (geometric) connected components of \mathcal{X}_P . Similarly, the fiber of $\pi' : \Gamma_{(X, \mathbb{P}')} \rightarrow \text{Trop}(X)$ over w is in canonical bijection with $C(\mathcal{X}'_{P'})$. By Lemma 3.5, both of these sets of connected components are in bijection with $C(\text{in}_w X)$; in fact,

we have a commutative diagram:

$$\begin{array}{ccc} C(\text{in}_w X) & \longrightarrow & (\pi')^{-1}(w) \\ \parallel & & \downarrow \\ C(\text{in}_w X) & \longrightarrow & \pi^{-1}(w) \end{array}$$

Thus the map $\Gamma_{(X, \mathbb{P}')} \rightarrow \Gamma_{(X, \mathbb{P})}$ is bijective, and is therefore a homeomorphism on the underlying topological spaces of $\Gamma_{(X, \mathbb{P}')} and $\Gamma_{(X, \mathbb{P})}$. $\square$$

In light of this proposition, we denote by Γ_X the underlying topological space of $\Gamma_{(X, \mathbb{P})}$ for *any* normal crossings pair (X, \mathbb{P}) . We think of Γ_X , together with its natural map to $\text{Trop}(X)$, as a “parameterized tropical variety”. Note that Γ_X inherits an integral affine structure by pullback from $\text{Trop}(X)$; more precisely, for any normal crossings pair (X, \mathbb{P}) , the map $\Gamma_X \rightarrow \text{Trop}(X)$ is linear on any polyhedron in $\Gamma_{(X, \mathbb{P})}$.

Note that for any $w \in \text{Trop}(X)$, the number of preimages of w in Γ_X is equal to the number of connected components of $\text{in}_w X$. Therefore, if Σ is a normal crossings decomposition of $\text{Trop}(X)$, and w is in the relative interior of a top dimensional cell P of Σ , then the number of preimages of w is equal to the *multiplicity* of P in $\text{Trop}(X)$. This suggests that we should give Γ_X the structure of a weighted polyhedral complex by giving every polyhedron on Γ_X weight one.

If we do this, then Γ_X satisfies a “balancing condition” analogous to the well-known balancing condition on $\text{Trop}(X)$. Fix a normal crossings decomposition Σ of $\text{Trop}(X)$, with corresponding normal crossings pair (X, \mathbb{P}) . Consider a polyhedron (P, Y) of $\Gamma_{(X, \mathbb{P})}$ of dimension $\dim X - 1$, and let $\{(P_i, Y_i)\}$ be the top dimensional polyhedra of $\Gamma_{(X, \mathbb{P})}$ containing (P, Y) .

Fix a point w with rational coordinates in the relative interior of P , and let V_P be the linear span, $\text{Span}(P - w)$. Similarly, for each P_i , let V_i be the positive span of $\text{Span}^+(P_i - w)$. Then V_i/V_P is a ray in \mathbb{R}^n/V_P ; this collection of rays is the fan attached to the toric variety \mathbb{P}_P . Let v_i be the smallest integer vector along the ray V_i/V_P .

Proposition 4.2. *The v_i ’s satisfy the “balancing property”:*

$$\sum_{(P_i, Y_i)} v_i = 0.$$

Proof. Torus-equivariant rational functions on \mathbb{P}_P correspond to lattice vectors u in the space $(\mathbb{R}^n/V_P)^*$ dual to \mathbb{R}^n/V_P . The valuation of u along the divisor of \mathbb{P}_P corresponding to v_i is simply $u(v_i)$.

Now restrict u to the curve \mathcal{X}_P . For any polyhedron P' of Σ containing P , \mathcal{X}_P intersects the boundary divisor $\mathbb{P}_{P'}$ in one point for each cell (P_i, Y_i) of $\Gamma_{(X, \mathbb{P})}$ with $P_i = P'$. The divisor of u is therefore equal to

$$\sum_{(P_i, Y_i)} u(v_i) Y_i,$$

as \mathcal{X}_P intersects each boundary divisor \mathbb{P}_{P_i} transversely. This divisor is a principal divisor and thus has degree zero. \square

We have thus attached to any schön subvariety X of \mathcal{T} , a canonical, multiplicity free parameterization by the topological space Γ_X . Moreover, this construction is

functorial: let \mathcal{T} and \mathcal{T}' be tori over \mathcal{O} , and let T and T' be their general fibers. Suppose we have schön subvarieties X and Y of T and T' , respectively, and a homomorphism of tori $T \rightarrow T'$ that takes X to Y . We then have a natural map $f : \text{Trop}(X) \rightarrow \text{Trop}(Y)$.

Proposition 4.3. *There is a natural map $\Gamma_X \rightarrow \Gamma_Y$ that makes the diagram*

$$\begin{array}{ccc} \Gamma_X & \longrightarrow & \Gamma_Y \\ \downarrow & & \downarrow \\ \text{Trop}(X) & \longrightarrow & \text{Trop}(Y) \end{array}$$

commute.

Proof. Let Σ' be a normal crossings decomposition of $\text{Trop}(Y)$. By proposition 3.9 we can find a normal crossings decomposition Σ of $\text{Trop}(X)$ such that the image of any cell of Σ under the map f is contained in a cell of Σ' . Let (X, \mathbb{P}) and (Y, \mathbb{P}') be the tropical pairs corresponding to Σ and Σ' , and let \mathcal{X} and \mathcal{Y} denote the associated tropical degenerations. Since each cell of Σ maps into a cell of Σ' , we obtain a map from \mathcal{X} to \mathcal{Y} extending the map $X \rightarrow Y$.

Now let P be a polyhedron in Σ , and P' be the polyhedron of Σ' containing the image of P . Then our map $\mathcal{X} \rightarrow \mathcal{Y}$ induces a map $\mathcal{X}_P \rightarrow \mathcal{Y}_{P'}$.

If (P, X_i) is a polyhedron of $\Gamma_{(X, \mathbb{P})}$, then by definition X_i is a connected component of \mathcal{X}_P . The image of X_i in $\mathcal{Y}_{P'}$ is contained in a connected component Y_i of $\mathcal{Y}_{P'}$. We can thus construct a map of polyhedral complexes

$$\Gamma_{(X, \mathbb{P})} \rightarrow \Gamma_{(Y, \mathbb{P}')}$$

that maps (P, X_i) to (P', Y_i) by the map $P \rightarrow P'$. The induced map $\Gamma_X \rightarrow \Gamma_Y$ on underlying topological spaces is clearly continuous and makes the diagram commute.

To see that it is independent of choices, let π_X and π_Y be the projections of Γ_X and Γ_Y to $\text{Trop}(X)$ and $\text{Trop}(Y)$ respectively. We then have canonical bijections between $\pi_X^{-1}(w)$ and $C(\text{in}_w X)$, and between $\pi_Y^{-1}(f(w))$ and $C(\text{in}_{f(w)} Y)$. The map $X \rightarrow Y$ induces a natural map $\text{in}_w X \rightarrow \text{in}_{f(w)} Y$, and the diagram

$$\begin{array}{ccc} C(\text{in}_w X) & \longrightarrow & \pi_X^{-1}(w) \\ \downarrow & & \downarrow \\ C(\text{in}_{f(w)} Y) & \longrightarrow & \pi_Y^{-1}(f(w)) \end{array}$$

commutes. As the left hand side is independent of the choices of Σ and Σ' , the result follows. \square

Remark 4.4. Although Proposition 4.3 is stated for maps $X \rightarrow Y$ that are *monomial morphisms* (i.e., that arise from morphisms of the ambient tori), we can avoid this issue if X and Y are intrinsically embedded. Recall that X is *very affine* if it can be embedded as a closed subscheme of a torus T . In this case (c.f. [T], section 3) there is an intrinsic torus T_X associated to X a *canonical* embedding of X in T_X . Moreover, if X and Y are very affine and $f : X \rightarrow Y$ is a morphism, there is a morphism of tori $T_X \rightarrow T_Y$ that induces f .

We also record, for later use, the following result relating the cohomology of Γ_X to that of $\text{Trop}(X)$:

Lemma 4.5. *Let X be schön, and let Σ be a normal crossings decomposition of $\mathrm{Trop}(X)$. Suppose that for each polyhedron P in Σ , \mathcal{X}_P is either connected or has dimension zero. Then the natural map*

$$H^r(\mathrm{Trop}(X), \mathbb{Z}) \rightarrow H^r(\Gamma_X, \mathbb{Z})$$

is an isomorphism for $0 \leq r < \dim X$, and an injection for $r = \dim X$.

Proof. Let (X, \mathbb{P}) be the normal crossings pair attached to Σ , so that $\Gamma_{(X, \mathbb{P})}$ is a triangulation of Γ . The polyhedra P in $\Gamma_{(X, \mathbb{P})}$ with $\dim P < \dim X$ are, by construction, in bijection with the polyhedra in Σ with $\dim P < \dim X$. Thus $\Gamma_{(X, \mathbb{P})}$ is obtained from Σ by adding additional top-dimensional cells; the result follows immediately. \square

5. WEIGHT FILTRATIONS AND THE WEIGHT SPECTRAL SEQUENCE

Our goal will be to relate the combinatorial structure of Γ_X to geometric invariants of X . The invariants that appear arise from Deligne's theory of weights, which we now summarize. Recall (c.f. [D], 1.2) that if F is a finite field of order q , a continuous l -adic representation ρ of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ has weight r if all the eigenvalues of the geometric Frobenius of F are algebraic integers α , all of whose Galois conjugates have complex absolute value equal to $q^{r/2}$. If A is a finitely generated \mathbb{Z} -algebra, an étale sheaf \mathcal{F} on $\mathrm{Spec} A$ has weight r if for each closed point s of $\mathrm{Spec} A$, the stalk \mathcal{F}_s has weight r when considered as a $\mathrm{Gal}(k(s)^{\mathrm{sep}}/k(s))$ -module.

Following Ito ([I], 2.2), we extend this definition to the case where F is a purely inseparable extension of a finitely generated extension of \mathbb{F}_p or \mathbb{Q} . For such F , one can find a finitely generated \mathbb{Z} -subalgebra A of F such that F is a purely inseparable extension of the field of fractions of A .

In this setting, a representation ρ of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ has weight r if there is an open subset U of $\mathrm{Spec} A$, and a smooth \mathcal{F} on U of weight r , such that ρ arises from \mathcal{F} by pullback to $\mathrm{Spec} F$. The Weil conjectures imply that for any proper smooth variety X over F , and any l prime to the characteristic of F , $H_{\mathrm{\acute{e}t}}^r(X_{F^{\mathrm{sep}}}, \mathbb{Q}_l)$ has weight r .

We henceforth assume that the residue field k of \mathcal{O} is a purely inseparable extension of a finitely generated extension of \mathbb{F}_p or \mathbb{Q} . We also fix an l prime to the characteristic of k .

Let G be the absolute Galois group of the field K . Then G admits a surjection $G \rightarrow \mathrm{Gal}(k^{\mathrm{sep}}/k)$, whose kernel is the inertia group I_K of K . If M is a G -module on which I acts through a finite quotient, there is a finite index subgroup H in G such that $H \cap I$ acts trivially on M . Thus $\mathrm{Gal}(k^{\mathrm{sep}}/k')$ acts on M for some finite extension k' of k . We say M is pure of weight r if it has weight r as a $\mathrm{Gal}(k^{\mathrm{sep}}/k')$ -module. Note that this is independent of k' .

The étale cohomology of a variety over K with semistable reduction has a filtration by subquotients which are pure in the above sense. More precisely, let \mathcal{X} be a proper scheme over \mathcal{O} , of relative dimension n , whose fiber X_K over $\mathrm{Spec} K$ is smooth and whose fiber \mathcal{X}_k over $\mathrm{Spec} k$ is a divisor with simple normal crossings. Then the Rapoport-Zink weight spectral sequence relates the étale cohomology of $X_{K^{\mathrm{sep}}}$ to the geometry of the special fiber \mathcal{X}_k . More precisely, let $\mathcal{X}_{k^{\mathrm{sep}}}^{(r)}$ denote the disjoint union of $(r+1)$ -fold intersections of irreducible components of $\mathcal{X}_{k^{\mathrm{sep}}}$; it is smooth of dimension $n-r$ over k^{sep} . We then have:

Theorem 5.1 ([RZ], Satz 2.10; see also [I]). *There is a spectral sequence:*

$$E_1^{-r, w+r} = \bigoplus_{s \geq \max(0, -r)} H_{\text{ét}}^{w-r-2s}(\mathcal{X}_{k^{\text{sep}}}^{(2s+r)}, \mathbb{Q}_l(-r-s)) \Rightarrow H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l).$$

Here $\mathbb{Q}_l(n)$ is the n th “Tate twist” of the constant sheaf \mathbb{Q}_l ; that is, it is the tensor product of \mathbb{Q}_l with the n th tensor power of the sheaf $\mathbb{Z}_l(1)$, where $\mathbb{Z}_l(1)$ is the inverse limit of the sheaves μ_{l^n} of l -power roots of unity. Note that $\mathbb{Z}_l(1)$ is pure of weight -2 .

The boundary maps of this spectral sequence are completely explicit, and can be described as follows: up to sign, they are direct sums of restriction maps

$$H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m)) \rightarrow H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m))$$

where Y is an irreducible component of $\mathcal{X}_{k^{\text{sep}}}^{(j)}$ and Y' is an irreducible component of $\mathcal{X}_{k^{\text{sep}}}^{(j+1)}$ contained in Y , or Gysin maps

$$H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m)) \rightarrow H_{\text{ét}}^{i+2}(Y, \mathbb{Q}_l(-m+1))$$

where Y and Y' are as above.

More precisely, each term $E_1^{p,q}$ is a direct sum of terms of the form $H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m))$ for some irreducible component Y of $\mathcal{X}_{k^{\text{sep}}}^{(j)}$. If Y' is an irreducible component of $\mathcal{X}_{k^{\text{sep}}}^{(j+1)}$, then we have:

- Whenever $H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p,q}$, and $H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p+1,q}$, then the corresponding direct summand of the boundary map $E_1^{p,q} \rightarrow E_1^{p+1,q}$ is (up to sign) the restriction

$$H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m)) \rightarrow H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m)).$$

- Whenever $H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m))$ is a direct summand of $E_1^{p,q}$, and $H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m+1))$ is a direct summand of $E_1^{p+1,q}$, then the corresponding direct summand of the boundary map $E_1^{p,q} \rightarrow E_1^{p+1,q}$ is (up to sign) the Gysin map

$$H_{\text{ét}}^i(Y', \mathbb{Q}_l(-m)) \rightarrow H_{\text{ét}}^{i+2}(Y, \mathbb{Q}_l(-m+1)).$$

We refer the reader to example 7.5 for a description of the weight spectral sequence in the case when X is a smooth curve.

Note that the term $E_1^{-r, w+r}$ of the weight spectral sequence is pure of weight $w+r$. As the only map between \mathbb{Q}_l -sheaves that are pure of different weights is the zero map, this implies that the weight spectral sequence degenerates at E_2 . Moreover, the successive quotients of the filtration on $H_{\text{ét}}^*(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ induced by the weight spectral sequence are pure. The filtration arising in this way is called the *weight filtration* on $H_{\text{ét}}^*(X_{K^{\text{sep}}}, \mathbb{Q}_l)$.

We say a G -module M is *mixed* if M admits an increasing G -stable filtration

$$\cdots \subset W_r M \subset W_{r+1} M \subset \cdots$$

such that $W_r M / W_{r-1} M$ has weight r for all r . (Such a filtration, if it exists, will be unique.) We say M is mixed of weights between r and r' if M is mixed and the quotients $W_i M / W_{i-1} M$ are nonzero only when $r \leq i \leq r'$. The above result shows that the cohomology of any scheme over K with semistable reduction is mixed. More generally, one has:

Theorem 5.2. (c.f. [I], 2.3) *Let X be a smooth, proper n -dimensional variety over K . Then $H_{\text{ét}}^r(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ is mixed of weights between $\max(0, 2r - 2n)$ and $\min(2n, 2r)$.*

Proof. If X has strictly semistable reduction, i.e., X is isomorphic to the general fiber of a scheme \mathcal{X} that is proper over \mathcal{O} , and whose special fiber is a divisor with simple normal crossings, then this follows from the weight spectral sequence. The general case follows by de Jong's theory of alterations [dJ]. \square

Proposition 5.3. *Let X be a smooth n -dimensional variety over K , and \overline{X} a compactification of X such that $\overline{X} - X$ is a divisor with simple normal crossings. Then for $r \leq n$, $H_{\text{ét}}^r(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ is mixed of weights between 0 and $2r$, and the natural map*

$$W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l) \rightarrow W_0 H_{\text{ét}}^r(X_{K^{\text{sep}}}, \mathbb{Q}_l)$$

is an isomorphism.

Proof. Let D be the divisor $\overline{X} \setminus X$, and let $\overline{D}_1, \dots, \overline{D}_r$ be its irreducible components. Let X_i be the open subset $X \setminus \{\overline{D}_1 \cup \dots \cup \overline{D}_i\}$. We proceed by induction on i ; the case $i = 0$ is clear.

Suppose the proposition is true for i . Define

$$D_i = \overline{D}_{i+1} \setminus \{\overline{D}_1 \cup \dots \cup \overline{D}_i\},$$

so that $X_i \setminus X_{i+1} = D_i$. By the inductive hypothesis the spaces $H_{\text{ét}}^r((X_i)_L, \mathbb{Q}_l)$ and $H_{\text{ét}}^r((D_i)_L, \mathbb{Q}_l)$ are mixed of weights between 0 and $2r$ for $r \leq n$. We have a Gysin sequence:

$$\begin{aligned} H_{\text{ét}}^{r-2}((D_i)_L, \mathbb{Q}_l(-1)) &\rightarrow H_{\text{ét}}^r((X_i)_L, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^r((X_{i+1})_L, \mathbb{Q}_l) \rightarrow \\ H_{\text{ét}}^{r-1}((D_i)_L, \mathbb{Q}_l(-1)) &\rightarrow \dots, \end{aligned}$$

and the first and last terms are mixed of weights between 2 and $2r$. It follows that $H_{\text{ét}}^r((X_{i+1})_L, \mathbb{Q}_l)$ is mixed of weights between 0 and $2r$ as required. We also obtain an isomorphism

$$W_0 H_{\text{ét}}^r((X_i)_L, \mathbb{Q}_l) \cong W_0 H_{\text{ét}}^r((X_{i+1})_L, \mathbb{Q}_l)$$

and hence by induction the desired isomorphism

$$W_0 H_{\text{ét}}^r(\overline{X}_L, \mathbb{Q}_l) \cong W_0 H_{\text{ét}}^r(X_L, \mathbb{Q}_l).$$

\square

6. COHOMOLOGY OF SCHÖN VARIETIES

The control that tropical geometry gives over the degenerations of schön subvarieties X of T has significant consequences on the level of cohomology. In particular the theory of vanishing cycles allows one to relate the étale cohomology of a nice tropical compactification of X to that of its tropical degeneration. When the degeneration is a divisor with simple normal crossings, this relationship is given by the Rapoport-Zink weight spectral sequence.

We apply this sequence in the setting of tropical geometry. Let X be schön. By Proposition 3.9 there is a polyhedral complex Σ , with support equal to $\text{Trop}(X)$ and corresponding toric scheme \mathbb{P} , such that the pair (X, \mathbb{P}) is tropical, the corresponding compactification \overline{X} of X is smooth with simple normal crossings boundary, and the special fiber of the corresponding tropical degeneration \mathcal{X} is a divisor with simple normal crossings.

The polyhedral complex $\Gamma_{(X, \mathbb{P})}$ encodes the combinatorics of the special fiber $\mathcal{X}_{k^{\text{sep}}}$. In particular $\mathcal{X}_{k^{\text{sep}}}$ is a union of smooth connected varieties \mathcal{X}_v , where v runs over the vertices of $\Gamma_{(X, \mathbb{P})}$. The varieties $\mathcal{X}_{v_1}, \dots, \mathcal{X}_{v_r}$ meet if and only if v_1, \dots, v_r are the vertices of a polyhedron in $\Gamma_{(X, \mathbb{P})}$. [Note that since \mathcal{X}_k is a simple normal crossings divisor, if Y_0, \dots, Y_r intersect in codimension r then they are the only irreducible components of $\mathcal{X}_{k^{\text{sep}}}$ containing their intersection.]

We have a natural map $\Gamma_{(X, \mathbb{P})} \rightarrow \Sigma$. Since Σ is a triangulation of $\text{Trop}(X)$, and $\Gamma_{(X, \mathbb{P})}$ is a triangulation of Γ_X , this induces a natural map

$$H^r(\text{Trop } X, \mathbb{Q}_l) \rightarrow H^r(\Gamma_X, \mathbb{Q}_l).$$

By the proof of Lemma 4.5, this map is an isomorphism if \mathcal{X}_P is connected for every polyhedron P in Σ , or (equivalently) if $\text{in}_w X$ is connected for every w in $\text{Trop}(X)$.

Theorem 6.1. *There is a natural isomorphism*

$$H^r(\Gamma_X, \mathbb{Q}_l) \cong W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l),$$

and hence a natural map

$$H^r(\text{Trop}(X), \mathbb{Q}_l) \rightarrow W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l).$$

This map is an isomorphism if \mathcal{X}_P is connected for every polyhedron P in Σ .

Proof. The bottom nonzero row of the E_1 term of the Rapoport-Zink spectral sequence (i.e., the $w = -r$ row) is the complex:

$$H_{\text{ét}}^0(\mathcal{X}_{k^{\text{sep}}}^{(0)}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^0(\mathcal{X}_{k^{\text{sep}}}^{(1)}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^0(\mathcal{X}_{k^{\text{sep}}}^{(2)}, \mathbb{Q}_l) \rightarrow \dots$$

in which the horizontal maps are restriction maps. This is simply the coboundary complex of the polyhedral complex formed by the bounded cells of $\Gamma_{(X, \mathbb{P})}$. This polyhedral complex is homotopy equivalent to $\Gamma_{(X, \mathbb{P})}$. We thus have a natural isomorphism

$$E_2^{r,0} \cong H^r(\Gamma_X, \mathbb{Q}_l).$$

□

Remark 6.2. Theorem 6.1 shows in particular that the space $W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$, which *a priori* depends on \overline{X} and thus a choice of Σ , in fact depends only on X and is *independent* of Σ . Proposition 5.3 establishes this directly on the level of cohomology.

The above results allow us to translate results about the cohomology of complete intersections in toric varieties into results about their tropicalizations. For instance:

Corollary 6.3. *Let X be a schön subvariety of T , and \mathbb{P}_K a smooth projective toric variety of T such that:*

- (1) *the closure Z of X in \mathbb{P}_K is a smooth complete intersection of ample divisors, and*
- (2) *the boundary $Z \setminus X$ is a divisor with simple normal crossings.*

Then $H^r(\Gamma_X, \mathbb{Q}_l) = 0$ for $1 \leq r < \dim X$.

Proof. By Proposition 3.9 and Theorem 6.1 there is a tropical pair (X, \mathbb{P}') , with corresponding compactification \overline{X} of X , such that $H^r(\Gamma_X, \mathbb{Q}_l)$ is isomorphic to $W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$. By Proposition 5.3 the latter is isomorphic to $W_0 H_{\text{ét}}^r(Z_{K^{\text{sep}}}, \mathbb{Q}_l)$.

Since Z is a complete intersection in \mathbb{P}_K , the Lefschetz hyperplane theorem shows that for $r < \dim X$, the restriction map

$$H_{\text{ét}}^r(\mathbb{P}_{K^{\text{sep}}}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^r(Z_{K^{\text{sep}}}, \mathbb{Q}_l)$$

is an isomorphism. But \mathbb{P}_K is a smooth toric variety, and hence has good reduction. The weight spectral sequence thus shows that $W_0 H_{\text{ét}}^r(PP_{K^{\text{sep}}}, \mathbb{Q}_l) = 0$ for $r > 0$. \square

Under more restrictive hypotheses on X , we can turn the above result into a result about the cohomology of $\text{Trop}(X)$. This will be the main goal of section 8.

7. MONODROMY

In many situations, the weight filtration has an alternative interpretation in terms of monodromy. Let X be a variety over K , and consider the base change $X_{K^{\text{sep}}}$ of X to K^{sep} . The group $\text{Gal}(K^{\text{sep}}/K)$ admits a map to $\text{Gal}(k^{\text{sep}}/k)$; the kernel is the inertia group I_K . The group I_K is profinite; if l is prime to the characteristic of k then the pro- l part $I_K^{(l)}$ of I_K is isomorphic to $\mathbb{Z}_l(1)$. (The Tate twist here refers to the fact that the quotient $\text{Gal}(k^{\text{sep}}/k)$ acts on $I_K^{(l)}$ by conjugation in the same way that it acts on the inverse limit of the roots of unity μ_{l^n} .)

The group $\text{Gal}(K^{\text{sep}}/K)$ acts on the étale cohomology $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ for any prime l . This action is quasi-unipotent, i.e. a subgroup of H of I_K of finite index acts unipotently on $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$. (And thus the action of H factors through $I_K^{(l)}$.) In particular there is a nilpotent map

$$N : H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l) \rightarrow H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l(-1))$$

called the *monodromy operator* such that for all $\sigma \in H$, σ acts on $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ by $\exp(t_l(\sigma)N)$, where t_l is the map $I_K \rightarrow I_K^{(l)} \cong \mathbb{Z}_l(1)$.

Now, if V is any finite dimensional vector space, with a nilpotent endomorphism N such that $N^r = 0$, then there is a unique increasing filtration $\{V_i\}$ on V such that:

- $V_r = V$,
- $V_{-r} = 0$,
- N maps V_i to V_{i-2} , and
- N^i induces an isomorphism $V_i/V_{i-1} \rightarrow V_{-i}/V_{-i-1}$.

(see [D] I, 1.7.2 for details.) We thus obtain a natural filtration, called the monodromy filtration, on $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$.

Remark 7.1. If V consists of a single Jordan block of dimension r , one sees easily that V_i/V_{i-1} is one-dimensional for $i \in \{r-1, r-3, \dots, -r+1\}$, and zero otherwise. Moreover, V_{r-1-2k} is the image of N^k for $0 \leq k \leq r-1$. It is thus straightforward to read off the filtration coming from an arbitrary V and N from a Jordan normal form for N . The filtration is independent of choices, even though the Jordan normal form of N is not.

When X has a semistable model, one can read the monodromy action on X off of the weight spectral sequence $E^{p,q}$. More precisely, there is a monodromy operator $N : E_1^{p,q} \rightarrow E_1^{p+2,q-2}(-1)$, which converges to the monodromy operator N on $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$. It is easily described: if $H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m))$ occurs as a direct

summand of $E_1^{p,q}$, and $H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m+1))$ occurs as a direct summand of $E_1^{p+2,q-2}$, then the corresponding direct summand of N is the identity

$$H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m)) \rightarrow H_{\text{ét}}^i(Y, \mathbb{Q}_l(-m+1))(-1).$$

All other direct summands of N are the zero map.

The following conjecture (the “weight-monodromy conjecture”) relates the weight filtration to the monodromy filtration in this situation:

Conjecture 7.2. *The top nonzero power of the monodromy operator:*

$$N^r : E_2^{-r,w+r} \rightarrow E_2^{r,w-r}$$

is an isomorphism for all r, w . In particular the weight filtration on $H_{\text{ét}}^i(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ E coincides (up to a shift in degree) with the monodromy filtration; that is,

$$H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r} / H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r-1} \cong W_{w-r} H_{\text{ét}}^w(X_{K^{\text{sep}}}, \mathbb{Q}_l).$$

The weight-monodromy conjecture is well-known to hold for curves and surfaces. If \mathcal{O} is an equal characteristic discrete valuation ring, it is a difficult theorem of Ito ([I], Theorem 1.1). It is open in general when \mathcal{O} has mixed characteristic.

For the remainder of this section we assume we are in a situation where Conjecture 7.2 holds. The following result, due to Speyer ([Sp2], Theorem 10.8) for curves, follows immediately:

Corollary 7.3. *Let $b_r(\Gamma_X)$ and $b_r(X)$ denote the r th Betti numbers of Γ_X and X , respectively. Then we have:*

$$b_r(\Gamma_X) \leq \frac{1}{r+1} b_r(X).$$

Proof. Theorem 6.1, together with the weight-monodromy conjecture, shows that $b_r(\Gamma_X)$ is the dimension of $H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)_{-r}$. The dimension of this piece of the monodromy filtration counts the number of Jordan blocks of size $r+1$ in a Jordan normal form for N acting on $H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$. In particular the dimension of the latter is at least $r+1$ times the dimension of the former. \square

Suppose X_K is an n -dimensional variety. There is a geometric interpretation of the action of the n th power of the monodromy operator on the middle-dimensional cohomology.

Proposition 7.4. *The n th power of the monodromy map acting on middle cohomology,*

$$N^n : H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_n / H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{n-1} \rightarrow H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{-n}(-n).$$

is the map

$$H_n(\Gamma_{(X, \mathbb{P})}, \mathbb{Q}_l)(-n) \rightarrow H^n(\Gamma_{(X, \mathbb{P})}, \mathbb{Q}_l)(-n)$$

induced from the “volume pairing” on the parameterizing complex $\Gamma_{X, \mathbb{P}}$ which takes a pair of (integral) n -dimensional cycles to the (oriented) lattice volume of their intersection.

Proof. The term $H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_n / H_{\text{ét}}^n(X_{K^{\text{sep}}}, \mathbb{Q}_l)_{n-1}$ is computed by the $(-n, n)$ -entry in the Rapoport-Zink spectral sequence. The n th row is:

$$H_{\text{ét}}^0(\mathcal{X}_{k^{\text{sep}}}^{(n)}, \mathbb{Q}_l)(-n) \rightarrow H_{\text{ét}}^2(\mathcal{X}_{k^{\text{sep}}}^{(n-1)}, \mathbb{Q}_l)(-n+1) \rightarrow \cdots \rightarrow H_{\text{ét}}^n(\mathcal{X}_{k^{\text{sep}}}^{(0)}, \mathbb{Q}_l)$$

where the horizontal map is the Gysin map of $\mathcal{X}^{(k)} \rightarrow \mathcal{X}^{(k-1)}$. Since each component of $\mathcal{X}^{(k)}$ is an $(n-k)$ -dimensional smooth variety, this is the chain complex

formed by the bounded cells of $\Gamma_{(X, \mathbb{P})}$. Consequently, $E_2^{-k, n} \cong H_k(\Gamma, \mathbb{Q}_l)(-k)$. Now,

$$N^n : E_2^{-n, n} \cong H_n(\Gamma, \mathbb{Q}_l)(-n) \rightarrow E_2^{0, 0}(-n) \cong H^n(\Gamma_{(X, \mathbb{P})}, \mathbb{Q}_l)(-n)$$

is induced from the identity map on $H_{\text{ét}}^0(\mathcal{X}_{k^{\text{sep}}}^{(n)}, \mathbb{Q}_l)$. In the language of homology and cohomology of $\Gamma_{(X, \mathbb{P})}$, it comes from the map $C_n(\Gamma_{(X, \mathbb{P})}) \rightarrow C^n(\Gamma_{(X, \mathbb{P})})$ taking a simplex F to the cocycle $\delta_F : C^n(\Gamma_{(X, \mathbb{P})}) \rightarrow \mathbb{Z}$ that is the indicator function of F . Consequently, if we view N^n as a bilinear pairing on $H_n(\Gamma, \mathbb{Q}_l)$, it is the volume pairing as every bounded top-dimensional cell of $\Gamma_{(X, \mathbb{P})}$ has volume 1. \square

Example 7.5. Suppose that X_K is a curve of genus g . Then $\mathcal{X}_{k^{\text{sep}}}^{(0)}$ is the normalization of $\mathcal{X}_{k^{\text{sep}}}$; it is a disjoint union of smooth curves C_i of genus g_i . On the other hand, $\mathcal{X}_{k^{\text{sep}}}^{(1)}$ is the set of singular points of $\mathcal{X}_{k^{\text{sep}}}$; each such point lies on exactly two of the C_i . The corresponding weight spectral sequence is nonzero only for $-1 \leq r \leq 1$ and $0 \leq w + r \leq 2$; it looks like:

$$\begin{array}{ccc} \bigoplus_{p \in \mathcal{X}_{k^{\text{sep}}}^{(1)}} \mathbb{Q}_l(-1) & \rightarrow & \bigoplus_i H_{\text{ét}}^2(C_i, \mathbb{Q}_l) & 0 \\ 0 & & \bigoplus_i H_{\text{ét}}^1(C_i, \mathbb{Q}_l) & 0 \\ 0 & & \bigoplus_i H_{\text{ét}}^0(C_i, \mathbb{Q}_l) & \rightarrow \bigoplus_{p \in \mathcal{X}_{k^{\text{sep}}}^{(1)}} \mathbb{Q}_l \end{array}$$

The sequence clearly degenerates at E_2 . The monodromy operator N is nonzero only from $E_1^{-1, 2}$ to $E_1^{1, 0}(-1)$; it is simply the identity map on

$$\bigoplus_{p \in \mathcal{X}_{k^{\text{sep}}}^{(1)}} \mathbb{Q}_l(-1).$$

We thus find that the middle quotient of the monodromy filtration on $H_{\text{ét}}^1(X_{K^{\text{sep}}}, \mathbb{Q}_l)$ is isomorphic to the direct sum of $H_{\text{ét}}^1(C_i, \mathbb{Q}_l)$, whereas the top and bottom quotients are isomorphic to $H_1(\Gamma, \mathbb{Q}_l)$, (resp. $H^1(\Gamma, \mathbb{Q}_l)$) where Γ is the dual graph of $\mathcal{X}_{k^{\text{sep}}}$. As above, the map $N : H_1(\Gamma, \mathbb{Q}_l) \rightarrow H^1(\Gamma, \mathbb{Q}_l)$ can be interpreted as the length pairing on $H_1(\Gamma, \mathbb{Q}_l)$.

This example has a more classical interpretation. If we let J be the Jacobian of \overline{X} , then the connected component of the identity in the special fiber of the Néron model of J over \mathcal{O} is an extension of an abelian variety by a torus; let χ be the character group of this torus. Then χ is naturally isomorphic to $H_1(\Gamma, \mathbb{Z})$. Moreover, one has a monodromy pairing $\chi \times \chi \rightarrow \mathbb{Z}$ (see [SGA] for details.) If one identifies χ with $H_1(\Gamma, \mathbb{Z})$, the resulting pairing on $H_1(\Gamma, \mathbb{Z})$ is precisely the length pairing.

To summarize:

Proposition 7.6. *If X is a schön open subset of a smooth proper curve \overline{X} over K , then the “length pairing” on Γ_X coincides with the monodromy pairing on the character group χ associated to the Jacobian of \overline{X} .*

This has connections to Mikhalkin’s construction of tropical Jacobians. Given a tropical curve Γ , which Mikhalkin interprets as a metric graph, the length pairing on Γ induces a map $\Gamma \rightarrow \text{Hom}(\Gamma, \mathbb{Z})$; Mikhalkin defines the tropical Jacobian of Γ to be the torus $\text{Hom}(\Gamma, \mathbb{R})/\Gamma$. This torus has a natural integral affine structure induced from that on $\text{Hom}(\Gamma, \mathbb{R})$. See [MZ] for details.

Mikhalkin’s definition is purely combinatorial, but has a nice interpretation in terms of the uniformization of abelian varieties: if J is the Jacobian of \overline{X} then

there is a pairing $\chi \times \chi \rightarrow \overline{K}^\times$ whose valuation is the monodromy pairing. This pairing gives an embedding of χ as a lattice in the torus $\mathrm{Hom}(\chi, \overline{K}^\times)$; the quotient $\mathrm{Hom}(\chi, \overline{K}^\times)/\chi$ is a rigid space isomorphic to J . If we “tropicalize” this space by taking valuations, we obtain the space $\mathrm{Hom}(\chi, \mathbb{R})/\chi$, where χ embeds into $\mathrm{Hom}(\chi, \mathbb{R})$ by the monodromy pairing. In particular the “tropicalization” of J is the tropical Jacobian of Γ_X .

The upshot is that- provided we are careful about what we mean by tropicalization- “tropicalization” commutes with taking Jacobians.

The following result of [KMM] is another easy consequence of this point of view:

Proposition 7.7 ([KMM], Theorem 6.4). *Let X be a schön open subset of an elliptic curve \overline{X} over K with potentially multiplicative reduction. Then the $H_1(\Gamma_X, \mathbb{Z})$ is isomorphic to \mathbb{Z} , and valuation of the j -invariant $j(\overline{X})$ is equal to $-a$, where a is the length of the unique cycle in Γ_X .*

Proof. Replacing \mathcal{O} with a suitable ramified extension we may assume that Γ_X is integral. Then \overline{X} has split multiplicative reduction. This base change scales both Γ_X and the valuation of $j(\overline{X})$ by the degree d of the extension. Now the tropical degeneration \mathcal{X} of X associated to Γ_X gives a model of \overline{X} whose special fiber is a chain of rational curves, of length equal to the lattice length a of the unique cycle in Γ_X . The conductor-discriminant formula ([Si], Theorem 11.1) then shows that the valuation of $j(X)$ is equal to $-a$. \square

In fact, it is easy to see that any smooth curve \overline{X} contains a schön open subset: take a semistable model of \overline{X} , embed it in $\mathbb{P}_{\mathcal{O}}^n$, let \mathcal{T} be the complement of $n+1$ hyperplanes in general position in $\mathbb{P}_{\mathcal{O}}^n$, and take $X = \mathcal{T} \cap \overline{X}$. Then the compactification \overline{X} of X in $\mathbb{P}_{\mathcal{O}}^n$ is tropical, and one verifies easily that the multiplication map is smooth. Thus the above result applies to all elliptic curves with potentially multiplicative reduction. Qu [Q] has shown that all smooth quasi-projective varieties over \mathbb{C} contain a schön open subset.

8. COMPLETE INTERSECTIONS

In the constant coefficient case, a (Zariski) general hyperplane section of a schön variety is schön. Unfortunately this is no longer true in the nonconstant coefficient case. For instance, let X_k be a singular hypersurface in T_k . Then any hypersurface X in T_K that reduces modulo π to X_k has $\mathrm{in}_{(0,\dots,0)} X = X_k$, and hence cannot be schön. The set of such X is a rigid analytic open subset of the projective space of hypersurfaces of fixed degree.

As this example suggests, to study loci of schön varieties in a nonconstant coefficient setting, one needs to work with the rigid analytic topology rather than the Zariski topology. (For the basics of the theory of rigid analytic spaces we refer the reader to [EKL] or [Sch]; we use very little here.)

To make precise the connection to rigid geometry, we first observe:

Lemma 8.1. *Let \mathbb{P} be a toric scheme, proper over \mathcal{O} , and let X be a subvariety of the open torus T in $\mathbb{P} \times_{\mathcal{O}} \mathrm{Spec} K$. Suppose that for all polyhedra P in the polyhedral complex Σ corresponding to P , the closure \mathcal{X} of X in \mathbb{P} intersects \mathbb{P}_P transversely. Then X is schön, and (X, \mathbb{P}') is a normal crossings pair, where \mathbb{P}' is the open subset of \mathbb{P} obtained by deleting all torus orbits that do not meet \mathcal{X} .*

Conversely, if X is schön and there exists a toric open subset \mathbb{P}' of \mathbb{P} such that (X, \mathbb{P}') is a normal crossings pair, then the closure of X intersects \mathbb{P}_P transversely for all polyhedra P in Σ .

Proof. Consider the multiplication map

$$m : \mathcal{T} \times \mathcal{X} \rightarrow \mathbb{P}'.$$

If y is a point in \mathbb{P}' in the torus orbit corresponding to a polyhedron P in the subcomplex Σ' of Σ corresponding to \mathbb{P}' , then the fiber over y is isomorphic to the product $\mathcal{X} \cap \mathbb{P}'_P$ with a torus. By assumption, this is smooth, so m has smooth fibers. The argument of [H], Lemma 2.6 then shows that m is smooth. It follows that X is schön and (X, \mathbb{P}') is a normal crossings pair. The converse is clear. \square

Note that the lemma implies that $\text{Trop}(X)$ will be equal to the support of Σ' for all such X . One can therefore use this result to study the space of schön subvarieties of a toric variety over K with a given tropicalization. We will not pursue this here, beyond a few straightforward observations.

Suppose \mathbb{P} is projective. Fix an X as in the lemma, and let $\text{Hilb}(\mathbb{P})$ be the Hilbert scheme over \mathcal{O} parameterizing subschemes of \mathbb{P} with the same Hilbert polynomial as the closure of X . Complex points of $\text{Hilb}(\mathbb{P})$ correspond to subschemes of the special fiber of \mathbb{P} ; those that meet each \mathbb{P}_P transversely form an open subset U_0 of $\text{Hilb}(\mathbb{P}) \times_{\mathcal{O}} \text{Spec } k$.

Now if y is a point of $\text{Hilb}(\mathbb{P})(\overline{K})$, then y corresponds to a subscheme X_y of the general fiber of \mathbb{P} over a finite extension of K . Then $X_y \cap T$ will satisfy the hypotheses of Lemma 8.1 if, and only if, y specializes to a point y_0 on the special fiber of $\text{Hilb}(\mathbb{P})$ that lies in U_0 . The set of points y that specialize to U_0 forms a “neighborhood of U_0 ” in the rigid analytic topology on $\text{Hilb}(\mathbb{P})$. More precisely, let $\text{Hilb}(\mathbb{P})^{\text{rig}}$ denote the rigid analytic space associated to the general fiber of $\text{Hilb}(\mathbb{P})$; then $\text{Hilb}(\mathbb{P})^{\text{rig}}$ is equipped with a “reduction mod π ” map

$$\text{Hilb}(\mathbb{P})^{\text{rig}} \rightarrow \text{Hilb}(\mathbb{P}) \times_{\mathcal{O}} \text{Spec } k.$$

The preimage of U_0 under this map is an admissible open subset U^{rig} of $\text{Hilb}(\mathbb{P})^{\text{rig}}$, and those $y \in \text{Hilb}(\mathbb{P})(\overline{K})$ such that $X_y \cap T$ satisfies the hypotheses of Lemma 8.1 are precisely the \overline{K} -points of U^{rig} .

If we restrict our attention to complete intersections, we can say more than this. In particular fix a projective toric scheme \mathbb{P} over \mathcal{O} , and ample line bundles L_1, \dots, L_s on \mathbb{P} . The space \mathcal{H} parameterizing tuples (D_1, \dots, D_s) such that for each i , D_i is an effective divisor in the linear system corresponding to L_i , and all the D_i ’s intersect transversely, is an open subset of a product of projective spaces over \mathcal{O} .

By Bertini’s theorem, the set of points in $\mathcal{H}(k)$ that correspond to divisors (D_1, \dots, D_s) in $\mathbb{P} \times_{\mathcal{O}} \text{Spec } k$ such that $D_1 \cap \dots \cap D_s$ intersects each stratum \mathbb{P}_P of \mathbb{P} transversely is an open dense subset U_0 of the special fiber of \mathcal{H} . The preimage of U_0 under the reduction map

$$\mathcal{H}^{\text{rig}} \rightarrow \mathcal{H} \times_{\mathcal{O}} \text{Spec } k$$

is a (necessarily nonempty) admissible open subset U^{rig} of \mathcal{H}^{rig} ; the points of U^{rig} correspond precisely to those complete intersections $D_1 \cap \dots \cap D_s$ whose intersection with T satisfies the conditions of Lemma 8.1.

Moreover, if (D_1, \dots, D_s) is a K -point of U^{rig} , and X is the corresponding complete intersection $D_1 \cap \dots \cap D_s \cap T$ in T , then for each polyhedron P in Σ , $\mathcal{X}_P = D_1 \cap \dots \cap D_s \cap \mathbb{P}_P$ is the intersection of ample divisors in the smooth toric variety \mathbb{P}_P , and is therefore either zero-dimensional or connected.

Lemma 4.5 and Theorem 6.1 now have immediate implications for the cohomology of $\text{Trop}(X)$:

Theorem 8.2. *Let (D_1, \dots, D_s) be a K -point of U^{rig} , and set*

$$X = D_1 \cap \dots \cap D_s \cap T.$$

Then $H^r(\text{Trop}(X), \mathbb{Q}_l)$ vanishes for $1 \leq r < \dim X$, and the natural map:

$$H^r(\text{Trop}(X), \mathbb{Q}_l) \rightarrow W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$$

is injective for $r = \dim X$.

Proof. The above discussion shows that X is schön and (X, \mathbb{P}) is a normal crossings pair. We thus apply Theorem 6.1 and Lemma 4.5 to see that the map

$$H^r(\text{Trop}(X), \mathbb{Q}_l) \rightarrow W_0 H_{\text{ét}}^r(\overline{X}_{K^{\text{sep}}}, \mathbb{Q}_l)$$

is an isomorphism for $0 \leq r < \dim X$ and injective for $r = \dim X$. On the other hand, \overline{X} is a complete intersection in the general fiber of the smooth toric variety $\mathbb{P} \times_{\mathcal{O}} \text{Spec } K$. The result thus follows from Corollary 6.3. \square

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